JOURNAL OF APPROXIMATION THEORY **40**, 391–392 (1984)

# Note

## On the Hermite Interpolation Polynomial

HANNU VÄLIAHO

*Department of Mathematics, University of Helsinki, Helsinki, Finland*

*Communicated by Oved Shisha*

Received April 29, 1982; revised May 3, 1983

THEOREM. *Let $f: [a, b] \to \mathbb{R}$ be sufficiently many times differentiable, and let $x_i \in [a, b]$, $m_i \in \mathbb{N}_+$, $i = 1,..., s$, be given. Then there exists a unique polynomial $r(x)$ of degree $< m := m_1 + \cdots + m_s$ (the Hermite interpolation polynomial) for which*

$$r^{(j)}(x_i) = f^{(j)}(x_i), \qquad i = 1,..., s, j = 0,..., m_i - 1. \tag{1}$$

There exist many proofs of this theorem, for example, Davis [1, Theorem 2.2.2] treats the problem in a more general setting. We present here an elementary inductive proof of the above theorem. The proof is constructive, i.e., it gives a method for constructing the Hermite interpolation polynomial.

*Proof of the Theorem.* Number the $m$ conditions (1) in the indicated order $t = 1,..., m$. The proof is by induction on $t$. For $t = 1$, the unique Hermite interpolation polynomial is $r(x) = f(x_1)$. We then assume the theorem to hold for $1,..., t - 1$ and establish it for $t$. Let $t > 1$ be of the form $t = m_1 + \cdots + m_{h-1} + k$, where $1 \leqslant k \leqslant m_h$. Any polynomial $r(x)$ of degree $< t$ can be expressed in the form

$$r(x) = p(x) + \lambda w(x), \tag{2}$$

where $w(x) = (x - x_1)^{m_1} \cdots (x - x_{h-1})^{m_{h-1}}(x - x_h)^{k-1}$, $p(x)$ is a polynomial of degree $< t - 1$, and $\lambda \in \mathbb{R}$. From (2),

$$r^{(j)}(x) = p^{(j)}(x) + \lambda w^{(j)}(x), \tag{3}$$

for any $j$. In order for $r(x)$ to be a Hermite interpolation polynomial of $f$ satisfying the first $t$ conditions there must be $r^{(j)}(x_i) = p^{(j)}(x_i) = f^{(j)}(x_i)$ for $i = 1,..., h - 1$; $j = 0,..., m_i - 1$ and, if $k > 1$, also for $i = h$; $j = 0,..., k - 2$,



0021-9045/84 $3.00



640/40/4-8

because $w^{(j)}(x_i)=0$ for these values of $i$ and $j$. Thus in (2), $p(x)$ must be the unique Hermite interpolation polynomial of $f$ satisfying the first $t-1$ conditions. It remains to determine $\lambda$ so that $r^{(k-1)}(x_h)=f^{(k-1)}(x_h)$. By means of (3) we derive uniquely,

$$\lambda=\frac{1}{w^{(k-1)}(x_h)}\,[f^{(k-1)}(x_h)-p^{(k-1)}(x_h)],$$

which is well defined because $w^{(k-1)}(x_h)\neq 0$. With this $\lambda$, (2) yields the required Hermite interpolation polynomial $r(x)$.

EXAMPLE. We construct the Hermite interpolation polynomial based on the data $x_1=1$, $x_2=2$; $m_1=4$, $m_2=3$; $f_{10}=1, f_{11}=-1, f_{12}=2, f_{13}=-6$; $f_{20}=3, f_{21}=4, f_{22}=-8$, where $f_{ij}=f^{(j)}(x_i)$, using an abbreviated method.

Denoting by $r_t(x)$ the Hermite interpolation polynomial satisfying the first $t$ conditions, we obtain, by Taylor's theorem, $r_4(x)=1-(x-1)+(x-1)^2-(x-1)^3$, and, according to the proof of the above theorem, $r_7(x)=r_4(x)+(x-1)^4q(x)$, where $q(x)$ is a polynomial of degree $<3$. We derive

$$r_7(x)=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4q(x),$$
$$r_7'(x)=-1+2(x-1)-3(x-1)^2+4(x-1)^3q(x)+(x-1)^4q'(x),$$
$$r_7''(x)=2-6(x-1)+12(x-2)^2q(x)+8(x-1)^3q'(x)+(x-1)^4q''(x).$$

With $x=2$, these equations yield in succession $q(2)=3$, $q'(2)=-6$, $q''(2)=8$. Thus $q(x)=3-6(x-2)+4(x-2)^2$, and finally,

$$r_7(x)=4x^6-38x^5+143x^4-273x^3+282x^2-152x+35.$$

## REFERENCES

1. P. J. DAVIS, "Interpolation and Approximation," Ginn (Blaisdell), Boston/New York/Toronto/London, 1963.
2. H. WERNER AND R. SCHABACK, "Praktische Mathematik II," Springer-Verlag, Berlin/Heidelberg/New York, 1972.